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Integration on quantum Euclidean space and sphere in N dimensions [1]

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Abstract

Invariant integrals of functions and forms over q - deformed Euclidean space and spheres in N dimensions are defined and shown to be positive definite, compatible with the star - structure and to satisfy a cyclic property involving the D - matrix of $SO_q(N)$. The definition is more general than the Gaussian integral known so far. Stokes theorem is proved with and without spherical boundary terms, as well as on the sphere.

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Disclaimer

1 Introduction

In recent years, there has been much interest in formulating physics and in particular field theory on quantized, i.e. noncommutative spacetime. One of the motivations is that if there are no more "points" in spacetime, such a theory should be well - behaved in the UV. The concept of integration on such a space can certainly be expected to be an essential ingredient. In the simplest case of the quantum plane, such an integral was first introduced by Wess and Zumino [1]; see also [2]. In the presumably more physical case of quantum Euclidean space [3], the Gaussian integration method was proposed by a number of authors [4, 6]. However, it is very tedious to calculate except in the simplest cases and its properties have never been investigated thoroughly; in fact, its domain of definition turns out to be rather small.

In this paper, we will give a different definition based on spherical integration in N dimensions and investigate its properties in detail. Although this idea has already appeared in the literature [7], it has not been developed very far. It turns out to be both simpler and more general than the Gaussian integral. We first show that there is a unique invariant integral over the quantum Euclidean sphere, and prove that it is positive definite and satisfies a cyclic property involving the D - matrix of $SO_q(N)$. The integral over quantum Euclidean space is then defined by radial integration, both for functions and N forms. It turns out not to be determined uniquely by rotation and translation invariance (=Stokes theorem) alone; it is unique after e.g. imposing a general scaling law. It is positive definite as well and thus allows to define a Hilbertspace of square - integrable functions, and satisfies the same cyclic property. The cyclic property also holds for the integral of N and $N-1$ - forms over spheres, which leads to a simple, truly noncommutative proof of Stokes theorem on Euclidean space with and without spherical boundary terms, as well as on the sphere. These proofs only work for $q \neq 1$, nevertheless they reduce to the classical Stokes theorem for $q \to 1$. This shows the power of noncommutative geometry. Obviously one would like to use this integral to define actions for field theories on such noncommutative spaces; this is work in progress.

Although only the case of quantum Euclidean space is considered, the general approach is clearly applicable to e.g. quantum Minkowski space as well.

2 Integral on the quantum sphere S_q^{N-1}

To establish the notation, we briefly summarize the definitions used in this paper, following Faddeev, Reshetikhin and Takhtadjan [3].

The (function algebra on the) quantum orthogonal group $O_q(N)$ (which is called $SO_q(N)$ in [3]) is the algebra generated by A^i_j modulo the relations

$$\hat{R}^{ik}_{mn} A^m_j A^n_l = A^i_n A^k_m \hat{R}^{nm}_{jl}, \tag{1}$$

$$g_{ij} A^i_k A^j_l = g_{kl}. \tag{2}$$

$SO_q(N)$ is obtained by further imposing

$$A^{i_1}_{j_1} \dots A^{i_N}_{j_N} \varepsilon_q^{j_1 \dots j_N} = \varepsilon_q^{i_1 \dots i_N} \tag{3}$$

using the fact that the quantum determinant is central, see e.g. [8].

The $\hat{R}$ - matrix decomposes into 3 projectors $R^{ij}_{kl} = (qP^+ - q^{-1}P^- + q^{1-N}P^0)^{ij}_{kl}$. The metric is determined by $(P^0)^{ij}_{kl} = \frac{q^2-1}{(q^N-1)(q^{2-N}+1)} g^{ij} g_{kl}$, where $g_{ik}g^{kj} = \delta^j_i$. In this paper, we assume q is real and positive. Then there is a star - structure (involution)

$$\overline{A^i_j} = g^{jm} A^l_m g_{li} \tag{4}$$

so that we really have $(S)O_q(N, I\!R)$, and the antipode can be written as

$$S(A^i_j) = \overline{A^j_i}. \tag{5}$$

Quantum Euclidean space [3] is generated by x^i with commutation relations

$$(P^-)^{ij}_{kl} x^k x^l = 0, \tag{6}$$

and the center is generated by 1 and $r^2 = g_{ij}x^i x^j$. The associated differentials satisfy $(P^+)^{ij}_{kl} dx^k dx^l = 0$ and $g_{ij}dx^i dx^j = 0$, i.e.

$$dx^i dx^j = -q\hat{R}^{ij}_{kl} dx^k dx^l. \tag{7}$$

The epsilon - tensor is then determined by the unique top - (N-) form

$$dx^{i_1}...dx^{i_N} = \varepsilon_q^{i_1...i_N} dx^1...dx^N \equiv \varepsilon_q^{i_1...i_N} d^N x. \tag{8}$$

The above relations are preserved under the coaction of $(S)O_q(N)$

$$\Delta(x^i) = A^i_j \otimes x^j \equiv x^i_{(1)} \otimes x^i_{(2)}, \tag{9}$$

in Sweedler - notation. The involution $\overline{x^i} = x^j g_{ji}$ is compatible with the left coaction of $(S)O_q(N, I\!R)$. One can also introduce derivatives which satisfy

$$(P^-)^{ij}_{kl} \partial^k \partial^l = 0, \tag{10}$$

$$\partial^i x^j = g^{ij} + q(\hat{R}^{-1})^{ij}_{kl} x^k \partial^l, \tag{11}$$

and

$$\partial^i dx^j = q^{-1}\hat{R}^{ij}_{kl} dx^k \partial^l, \quad x^i dx^j = q\hat{R}^{ij}_{kl} dx^k x^l. \tag{12}$$

This represents one possible choice. For more details, see e.g. [9]. Finally, the quantum sphere S_q^{N-1} is generated by $t^i = x^i/r$, which satisfy $g_{ij}t^i t^j = 1$.

We first define a (complex - valued) integral $< f(t) >_t$ of a function $f(t)$ over S_q^{N-1}. It should certainly be invariant under $O_q(N)$, which means

$$A^{i_1}_{j_1}...A^{i_n}_{j_n} < t^{j_1}...t^{j_n} >_t = < t^{i_1}...t^{i_n} >_t . \tag{13}$$

Of course, it has to satisfy

$$g_{i_l i_{l+1}} < t^{i_1}...t^{i_n} >_t = < t^{i_1}...t^{i_{l-1}} t^{i_{l+2}}...t^{i_n} >_t \quad \text{and} \quad (P^-)^{i_l i_{l+1}}_{j_l j_{l+1}} < t^{j_1}...t^{j_n} >_t = 0 \tag{14}$$

We require one more property, namely that $I^{i_1...i_n} \equiv < t^{i_1}...t^{i_n} >_t$ is analytic in $(q-1)$, i.e. its Laurent series in $(q-1)$ has no negative terms (we can then assume that the classical limit $q = 1$ is nonzero). These properties in fact determine the spherical integral uniquely: for n odd, one should define $< t^{i_1}...t^{i_n} >_t = 0$, and

Proposition 2.1 *For even n, there exists (up to normalization) one and only one tensor $I^{i_1\dots i_n} = I^{i_1\dots i_n}(q)$ analytic in $(q-1)$ which is invariant under $O_q(N)$*

$$A^{i_1}_{j_1}\dots A^{i_n}_{j_n} I^{j_1\dots j_n} = I^{i_1\dots i_n} \tag{15}$$

and symmetric,

$$(P^-)^{i_l i_{l+1}}_{j_l j_{l+1}} I^{j_1\dots j_n} = 0 \tag{16}$$

for any l. It can be normalized such that

$$g_{i_l i_{l+1}} I^{i_1\dots i_n} = I^{i_1\dots i_{l-1} i_{l+2}\dots i_n} \tag{17}$$

for any l. $I^{ij} \propto g^{ij}$.

An explicit form is e.g. $I^{i_1\dots i_n} = \lambda_n(\Delta^{n/2} x^{i_1}\dots x^{i_n})$, where $\Delta = g_{ij}\partial^i\partial^j$ is the Laplacian (in either of the 2 possible calculi), and λ_n is analytic in $(q-1)$. For $q = 1$, they reduce to the classical symmetric invariant tensors.

Proof The proof is by induction on n. For $n = 2$, g^{ij} is in fact the only invariant symmetric (and analytic) such tensor.

Assume the statement is true for n, and suppose I_{n+2} and I'_{n+2} satisfy the above conditions. Using the uniqueness of I_n, we have (in shorthand - notation)

$$\begin{aligned} g_{12} I_{n+2} &= f(q-1) I_n & (18)\\ g_{12} I'_{n+2} &= f'(q-1) I_n & (19) \end{aligned}$$

where the $f(q-1)$ are analytic, because the left - hand sides are invariant, symmetric and analytic. Then $J_{n+2} = f' I_{n+2} - f I'_{n+2}$ is symmetric, analytic, and satisfies $g_{12} J_{n+2} = 0$. It remains to show that $J = 0$.

Since J is analytic, we can write

$$J^{i_1\dots i_n} = \sum_{k=n_0}^{\infty} (q-1)^k J^{i_1\dots i_n}_{(k)}. \tag{20}$$

$(q-1)^{-n_0} J^{i_1\dots i_n}$ has all the properties of J and has a well-defined, nonzero limit as $q \to 1$; so we may assume $J_{(0)} \neq 0$.

Now consider invariance,

$$J^{i_1\dots i_n} = A^{i_1}_{j_1}\dots A^{i_n}_{j_n} J^{j_1\dots j_n}. \tag{21}$$

This equation is valid for all q, and we can take the limit $q \to 1$. Then A^i_j generate the commutative algebra of functions on the classical Lie group $O(N)$, and J becomes $J_{(0)}$, which is just a classical tensor. Now $(P^-)^{i_l i_{l+1}}_{j_l j_{l+1}} J^{j_1\dots j_n} = 0$ implies that $J_{(0)}$ is symmetric for neighboring indices, and therefore it is completely symmetric. With $g_{12} J = 0$, this implies that $J_{(0)}$ is totally traceless (classically!). But there exists no totally symmetric traceless invariant tensor for $O(N)$. This proves uniqueness. In particular, $I^{i_1\dots i_n} = \lambda_n(\Delta^{n/2} x^{i_1}\dots x^{i_n})$ obviously satisfies the assumptions of the proposition; it is analytic, because in evaluating

the Laplacians, only the metric and the $\hat{R}$ - matrix are involved, which are both analytic. Statement (17) now follows because x^2 is central. $\square$

Such invariant tensors have also been considered in [4] (where they are called S), as well as the explicit form involving the Laplacian. Our contribution is a self - contained proof of their uniqueness. So $< t^{i_1} ... t^{i_n} >_t \equiv I^{i_1 \cdots i_n}$ for even n (and 0 for odd n) defines the unique invariant integral over S_q^{N-1}, which thus coincides with the definition given in [7].

From now on we only consider $N \geq 3$ since for $N = 1, 2$, Euclidean space is undeformed. The following lemma is the origin of the cyclic properties of invariant tensors. For quantum groups, the square of the antipode is usually not 1. For $(S)O_q(N)$, it is generated by the D - matrix: $S^2 A^i_j = D^i_l A^l_k (D^{-1})^k_j$ where $D^i_l = g^{ik} g_{lk}$ (note that D also defines the quantum trace). Then

Lemma 2.2 *For any invariant tensor $J^{i_1 \cdots i_n} = A^{i_1}_{j_1} ... A^{i_n}_{j_n} J^{j_1 \cdots j_n}$, $D^{i_1}_{l_1} J^{i_2 \ldots l_1}$ is invariant too:*

$$A^{i_1}_{j_1} ... A^{i_n}_{j_n} D^{j_1}_{l_1} J^{j_2 \ldots l_1} = D^{i_1}_{l_1} J^{i_2 \ldots l_1} \tag{22}$$

Proof From the above, (22) amounts to

$$(S^{-2} A^{i_1}_{j_1}) A^{i_2}_{j_2} ... A^{i_n}_{j_n} J^{j_2 \cdots j_n j_1} = J^{i_2 \cdots i_n i_1}. \tag{23}$$

Multiplying with $S^{-1} A^{i_0}_{i_1}$ from the left and using $S^{-1}(ab) = (S^{-1}b)(S^{-1}a)$ and $(S^{-1} A^{i_1}_{j_1}) A^{i_0}_{i_1} = \delta^{i_0}_{j_1}$, this becomes

$$A^{i_2}_{j_2} ... A^{i_n}_{j_n} J^{j_2 \cdots j_n i_0} = S^{-1} A^{i_0}_{i_1} J^{i_2 \cdots i_n i_1}. \tag{24}$$

Now multiplying with $A^{l_0}_{i_0}$ from the right, we get

$$A^{i_2}_{j_2} ... A^{i_n}_{j_n} A^{l_0}_{i_0} J^{j_2 \cdots j_n i_0} = \delta^{l_0}_{i_1} J^{i_2 \cdots i_n i_1}. \tag{25}$$

But the (lhs) is just $J^{i_2 \cdots i_n l_0}$ by invariance and thus equal to the (rhs). $\square$

We can now show a number of properties of the integral over the sphere:

Theorem 2.3

$$\begin{aligned} \overline{< f(t) >_t} \quad &= \quad < \overline{f(t)} >_t & (26) \\ < \overline{f(t)} f(t) >_t \quad &\geq \quad 0 & (27) \\ < f(t) g(t) >_t \quad &= \quad < g(t) f(Dt) >_t & (28) \end{aligned}$$

where $(Dt)^i = D^i_j t^j$. The last statement follows from

$$I^{i_1 \cdots i_n} = D^{i_1}_{j_1} I^{i_2 \cdots i_n j_1}. \tag{29}$$

Proof For (26), we have to show that $I^{j_n...j_1}g_{j_ni_n}...g_{j_1i_1} = I^{i_1...i_n}$. Using the uniqueness of I, it is enough to show that $I^{j_n...j_1}g_{j_ni_n}...g_{j_1i_1}$ is invariant, symmetric and normalized as I. So first,

$$\begin{aligned} A^{i_1}_{j_1}...A^{i_n}_{j_n}\left(I^{k_n...k_1}g_{k_nj_n}...g_{k_1j_1}\right) &= g_{l_1i_1}...g_{l_ni_n}\overline{A^{l_n}_{k_n}...A^{l_1}_{k_1}}I^{k_n...k_1} \\ &= \overline{A^{l_n}_{k_n}...A^{l_1}_{k_1}I^{k_n...k_1}}g_{l_1i_1}...g_{l_ni_n} \\ &= \left(I^{l_n...l_1}g_{l_ni_n}...g_{l_1i_1}\right). \end{aligned} \tag{30}$$

We have used that I is real (since g^{ij} and $\hat{R}$ are real), and $A^{i_1}_{j_1}g_{k_1j_1} = g_{l_1i_1}\overline{A^{l_1}_{k_1}}$. The symmetry condition (16) follows from standard compatibility conditions between $\hat{R}$ and g^{ij}, and the fact that $\hat{R}$ is symmetric. The correct normalization can be seen easily using $g^{ij} = g_{ij}$ for q - Euclidean space.

To show positive definiteness (27), we use the observation made by [3] that

$$t^i \to A^i_j u^j \tag{31}$$

with $u^j = u_1\delta^j_1 + u_N\delta^j_N$ is an embedding $S^{N-1}_q \to Fun(O_q(N))$ for $u_1u_N = (q^{(N-2)/2} + q^{(2-N)/2})^{-1}$, since $(P^-)^{ij}_{kl}u^ku^l = 0$ and $g_{ij}u^iu^j = 1$. In fact, this embedding also respects the star - structure if one chooses $u_N = u_1q^{1-N/2}$ and real. Now one can write the integral over S^{N-1}_q in terms of the Haar - measure on the compact quantum group $O_q(N, I\!R)$ [10, 11]. Namely,

$$< t^{i_1}...t^{i_n} >_t = < A^{i_1}_{j_1}...A^{i_n}_{j_n} >_A u^{j_1}...u^{j_n} \equiv < A^{\underline{i}}_{\underline{j}} >_A u^{\underline{j}}, \tag{32}$$

(in short notation) since the Haar - measure $<>_A$ is left (and right) - invariant $< A^{\underline{i}}_{\underline{j}} >_A = A^{\underline{i}}_{\underline{k}} < A^{\underline{k}}_{\underline{j}} >_A = < A^{\underline{i}}_{\underline{k}} >_A A^{\underline{k}}_{\underline{j}}$ and analytic, and the normalization condition is satisfied as well. Then $< \overline{t^{\underline{i}}}t^{\underline{j}} >_t = < \overline{A^{\underline{i}}_{\underline{k}}}A^{\underline{j}}_{\underline{r}} >_A u^{\underline{k}}u^{\underline{r}}$ and for $f(t) = \sum f_{\underline{i}}t^{\underline{i}}$ etc.,

$$\begin{aligned} < \overline{f(t)}g(t) >_t &= \overline{f_{\underline{i}}}g_{\underline{j}} < \overline{A^{\underline{i}}_{\underline{k}}}A^{\underline{j}}_{\underline{r}} >_A u^{\underline{k}}u^{\underline{r}} = < \overline{\left(f_{\underline{i}}A^{\underline{i}}_{\underline{k}}u^{\underline{k}}\right)}\left(g_{\underline{j}}A^{\underline{j}}_{\underline{r}}u^{\underline{r}}\right) >_A \\ &= < \overline{f(Au)}g(Au) >_A . \end{aligned} \tag{33}$$

This shows that the integral over S^{N-1}_q is positive definite, because the Haar - measure over compact quantum groups is positive definite [10], cp. [12].

Finally we show the cyclic property (29). (28) then follows immediately. For $n = 2$, the statement is obvious: $g^{ij} = D^i_k g^{jk}$.
Again using a shorthand - notation, define

$$J^{12...n} = D_1 I^{23...n1}. \tag{34}$$

Using the previous proposition, we only have to show that J is symmetric, invariant, analytic and properly normalized. Analyticity is obvious. The normalization follows immediately by induction, using the property shown in proposition (2.1). Invariance of J follows from the

above lemma. It remains to show that J is symmetric, and the only nontrivial part of that is $(P^-)_{12}J^{12\ldots n} = 0$. Define

$$\tilde{J}^{12\ldots n} = (P^-)_{12}J^{12\ldots n}, \tag{35}$$

so $\tilde{J}$ is invariant, antisymmetric and traceless in the first two indices (12), symmetric in the remaining indices (we will say that such a tensor has the ISAT property), and analytic. It is shown below that there is no such $\tilde{J}$ for $q = 1$ (and $N \geq 3$). Then as in proposition (2.1), the leading term of the expansion of $\tilde{J}$ in $(q-1)$ is classical and therefore vanishes, which proves that $\tilde{J} = 0$ for any q.

So from now on $q = 1$. We show by induction that $\tilde{J} = 0$. This is true for $n = 2$: there is no invariant antisymmetric traceless tensor with 2 indices (for $N \geq 3$). Now assume the statement is true for n even, and that $\tilde{J}^{12\ldots(n+2)}$ has the ISAT property. Define

$$K^{12\ldots n} = g_{(n+1),(n+2)}\tilde{J}^{12\ldots(n+2)}. \tag{36}$$

K has the ISAT property, so by the induction assumption

$$K = 0. \tag{37}$$

Define

$$M^{145\ldots(n+2)} = g_{23}\tilde{J}^{12\ldots(n+2)} = \mathcal{S}_{14}M^{145\ldots(n+2)} + \mathcal{A}_{14}M^{145\ldots(n+2)} \tag{38}$$

where $\mathcal{S}$ and $\mathcal{A}$ are the classical symmetrizer and antisymmetrizer. Again by the induction assumption, $\mathcal{A}_{14}M^{145\ldots(n+2)} = 0$ (it satisfies the ISAT property). This shows that M is symmetric in the first two indices $(1,4)$. Together with the definition of M, this implies that M is totally symmetric. Further, $g_{14}M^{145\ldots(n+2)} = g_{14}g_{23}\tilde{J}^{12\ldots(n+2)} = 0$ because $\tilde{J}$ is antisymmetric in $(1,2)$. But then M is totally traceless, and as in proposition (2.1) this implies $M = 0$. Together with (37) and the ISAT property of $\tilde{J}$, it follows that $\tilde{J}$ is totally traceless. So $\tilde{J}$ corresponds to a certain Young tableaux, describing a larger - than - one dimensional irreducible representation of $O(N)$. However, $\tilde{J}$ being invariant means that it is a trivial one - dimensional representation. This is a contradiction and proves $\tilde{J} = 0$.

□

Property (27)[1] in particular means that one can now define the Hilbertspace of square - integrable functions on S_q^{N-1}. The same will be true for the integral on the entire Quantum Euclidean space.

The cyclic property (28) is a strong constraint on $I^{i_1 \ldots i_n}$ and could actually be used to calculate it recursively, besides its obvious interest in its own.

3 Integral over quantum Euclidean space

It is now easy to define an integral over quantum Euclidean space. Since the invariant length $r^2 = g_{ij}x^ix^j$ is central, we can use its square root r as well, and write any function

[1] as was pointed out to me by G. Fiore, positivity is also implied by results in [4]

on quantum Euclidean space in the form $f(x^i) = f(t^i, r)$. We then define its integral to be

$$< f(x) >_x = << f(t, r) >_t (r) \cdot r^{N-1} >_r, \tag{39}$$

where $< f(t, r) >_t (r)$ is a classical, analytic function in r, and $< g(r) >_r$ is some linear functional in r, to be determined by requiring Stokes theorem. It is essential that this radial integral $< g(r) >_r$ is really a functional of the *analytic continuation of g(r)* to a function on the (positive) real line. Only then one obtains a large class of integrable functions, and this concept of integration over the entire space agrees with the classical one. This is also the reason why the Gaussian integration procedure suggested e.g. in [4, 6] works only for a very small class of functions of the form $p(x)g_\alpha(x)$ where $g_\alpha(x)$ is a Gaussian and $p(x)$ is a certain class of power - series. It diverges as soon as $p(x)$ has a finite radius of "convergence", such as e.g. $1/(r^2 + 1)$; even classically, one cannot do such integrals term by term. This problem does not occur for the spherical integral.

It will turn out that Stokes theorem e.g. in the form $< \partial_i f(x) >_x = 0$ holds if and only if the radial integral satisfies the scaling property

$$< g(qr) >_r = q^{-1} < g(r) >_r . \tag{40}$$

This can be shown directly; we will instead give a more elegant proof later. This scaling property is obviously satisfied by an arbitrary superposition of Jackson - sums,

$$< f(r) >_r = \int_1^q dr_0 \mu(r_0) \sum_{n=-\infty}^{\infty} f(q^n r_0) q^n \tag{41}$$

with arbitrary (positive) "weight" $\mu(r_0) > 0$. If $\mu(r_0)$ is a delta - function, this is simply a Jackson - sum; for $\mu(r_0) = 1$, one obtains the classical radial integration

$$< f(r) r^{N-1} >_r = \int_1^q dr_0 q^n (q^n r_0)^{N-1} \sum_{n=-\infty}^{\infty} f(q^n r_0) = \int_0^\infty dr r^{N-1} f(r). \tag{42}$$

For Gaussian - integrable functions, all of these uncountable choices are equivalent (and of course agree with that definition). This is in general not true for functions integrable in this radial sense, i.e. for which the above is finite, and shows again that this class is indeed larger. The classical integral over r however is the unique choice for which the scaling property (40) holds for any positive real number, not just for powers of q.

The properties of the integral over S_q^{N-1} generalize immediately to the Euclidean case:

Theorem 3.4

$$\overline{< f(x) >_x} = < \overline{f(x)} >_x \tag{43}$$

$$< \overline{f(x)} f(x) >_x \geq 0 \tag{44}$$

$$< f(x) g(x) >_x = < g(x) f(Dx) >_x, \tag{45}$$

and

$$< f(qx) >_x = q^{-N} < f(x) >_x \tag{46}$$

if and only if (40) holds.

Proof Immediately from theorem (2.3), (40) and (39), using $Dr = r$ and $\mu(r_0) > 0$. $\square$

(43) and (46) were already known for the special case of the Gaussian integral [4][2].

4 Integration of forms

It turns out to be very useful to consider not only integrals over functions, but also over forms, just like classically. As was mentionned before, there exists a unique N - form $dx^{i_1}...dx^{i_N} = \varepsilon_q^{i_1...i_N} d^N x$, and we define

$$\int_x d^N x f(x) = < f(x) >_x, \tag{47}$$

i.e. we first commute $d^N x$ to the left, and then take the integral over the function on the right. Then the two statements of Stokes theorem $< \partial_i f(x) >_x = 0$ and $\int_x d\omega_{N-1} = 0$ are equivalent.

The following observation by Bruno Zumino [13] will be very useful: there is a one - form

$$\omega = \frac{q^2}{(q+1)r^2} d(r^2) = q\frac{1}{r}dr = dr\frac{1}{r} \tag{48}$$

where $r dx^i = q dx^i r$, which generates the calculus on quantum Euclidean space by

$$[\omega, f]_\pm = (1-q)df \tag{49}$$

for any form f with the appropriate grading. It satisfies

$$d\omega = \omega^2 = 0. \tag{50}$$

We define the integral of a N - form over the sphere $r \cdot S_q^{N-1}$ with radius r by

$$\int\limits_{r\cdot S_q^{N-1}} d^N x f(x) = \omega r^N < f(x) >_t = dr r^{N-1} < f(x) >_t, \tag{51}$$

which is a one - form in r, as classically. It satisfies

$$\int\limits_{r\cdot S_q^{N-1}} q^N d^N x f(qx) = \int\limits_{qr\cdot S_q^{N-1}} d^N x f(x) \tag{52}$$

where $(drf(r))(qr) = qdrf(qr)$. Now defining $\int_r drg(r) =< g(r) >_r$, (47) can be written as

$$\int_x d^N x f(x) = \int_r (\int\limits_{r\cdot S_q^{N-1}} d^N x f(x)). \tag{53}$$

[2]it was pointed out to me by G. Fiore that positivity for certain classes of functions was shown in [5]

The scaling property (40), i.e. $\int_x d^N x f(qx) = q^{-N} \int_x d^N x f(x)$ holds if and only if the radial integrals satisfies

$$\int_r dr f(qr) = q^{-1} \int_r dr f(r). \tag{54}$$

We can also define the integral of a $(N-1)$ form $\alpha_{N-1}(x)$ over the sphere with radius r:

$$\int\limits_{r \cdot S_q^{N-1}} \alpha_{N-1} = \omega^{-1}(\int\limits_{r \cdot S_q^{N-1}} \omega \alpha_{N-1}). \tag{55}$$

The ω^{-1} simply cancels the explicit ω in (51), and it reduces to the correct classical limit for $q = 1$.

The epsilon - tensor satisfies the cylic property:

Proposition 4.5

$$\varepsilon_q^{i_1 \dots i_N} = (-1)^{N-1} D^{i_i}_{j_1} \varepsilon_q^{i_2 \dots i_N j_1}. \tag{56}$$

Proof Define

$$\kappa^{12 \dots N} = (-1)^{N-1} D^1 \varepsilon_q^{23 \dots N1} \tag{57}$$

in shorthand - notation again. Lemma (2.2) shows that κ is invariant. $\kappa^{12 \dots N}$ is traceless and (q -) antisymmetric in (23...N). Now $g_{12} \kappa^{12 \dots N} = 0$ because there exists no invariant, totally antisymmetric traceless tensor with $(N-2)$ indices for $q = 1$, so by analyticity there is none for arbitrary q. Similarly from the theory of irreducible representations of $SO(N)$ [14], ${P^+}_{12} \kappa^{12 \dots N} = 0$ where P^+ is the q - symmetrizer, $1 = P^+ + P^- + P^0$. Therefore $\kappa^{12 \dots N}$ is totally antisymmetric and traceless (for neighboring indices), invariant and analytic. But there exists only one such tensor up to normalization (which can be proved similarly), so $\kappa^{12 \dots N} = f(q) \varepsilon_q^{12 \dots N}$. It remains to show $f(q) = 1$. By repeating the above, one gets $\varepsilon_q^{12 \dots N} = (f(q))^N (\det D) \varepsilon_q^{12 \dots N}$ (here $12 \dots N$ stands for the *numbers* 1,2,...,N), and since $\det D = 1$, it follows $f(q) = 1$ (times a N-th root of unity, which is fixed by the classical limit). $\square$

Now consider a k - form $\alpha_k(x) = dx^{i_1} \dots dx^{i_k} f_{i_1 \dots i_k}(x)$ and a $(N-k)$ - form $\beta_{N-k}(x)$. Then the following cyclic property for the integral over forms holds:

Theorem 4.6

$$\int\limits_{r \cdot S_q^{N-1}} \alpha_k(x) \beta_{N-k}(x) = (-1)^{k(N-k)} \int\limits_{q^{-k} r \cdot S_q^{N-1}} \beta_{N-k}(x) \alpha_k(q^N D x) \tag{58}$$

where $\alpha_k(q^N D x) = (q^N D dx)^{i_1} \dots (q^N D dx)^{i_k} f_{i_1 \dots i_k}(q^N D x)$.

In particular, when α_k *and* β_{N-k} *are forms on* S_q^{N-1}*, i.e. they involve only* $dx^i \frac{1}{r}$ *and* t^i*, then*

$$\int\limits_{S_q^{N-1}} \alpha_k(t) \beta_{N-k}(t) = (-1)^{k(N-k)} \int\limits_{S_q^{N-1}} \beta_{N-k}(t) \alpha_k(D t). \tag{59}$$

On Euclidean space,

$$\int_x \alpha_k(x)\beta_{N-k}(x) = (-1)^{k(N-k)}\int_x \beta_{N-k}(x)\alpha_k(q^N Dx) \tag{60}$$

if and only if (54) holds.

Notice that on the sphere, $d^N x f(t) = f(t) d^N x$.
Proof We only have to show that

$$\int\limits_{r\cdot S_q^{N-1}} f(x)d^N x g(x) = \int\limits_{r\cdot S_q^{N-1}} d^N x g(x) f(q^N Dx) \tag{61}$$

and

$$\int\limits_{r\cdot S_q^{N-1}} dx^i \beta_{N-1}(x) = (-1)^{N-1}\int\limits_{q^{-1}r\cdot S_q^{N-1}} \beta_{N-1}(x)(q^N Ddx)^i. \tag{62}$$

(61) follows immediately from (28) and $x^i d^N x = d^N x q^N x^i$.

To see (62), we can assume that $\beta_{N-1}(x) = dx^{i_2}...dx^{i_N} f(x)$. The commutation relations $x^i dx^j = q\hat{R}^{ij}_{kl} dx^k x^l$ are equivalent to

$$\begin{aligned} f(q^{-1}x)dx^j &= \mathcal{R}((dx^j)_{(a)} \otimes f_{(1)})(dx^j)_{(b)}(f(x))_{(2)} \\ &= (dx^j \triangleleft \mathcal{R}^1)(f(x) \triangleleft \mathcal{R}^2) \end{aligned} \tag{63}$$

where $\mathcal{R} = \mathcal{R}^1 \otimes \mathcal{R}^2$ is the universal $\mathcal{R}$ for $SO_q(N)$, using its quasitriangular property and $\mathcal{R}(A^j_k \otimes A^i_l) = \hat{R}^{ij}_{kl}$. $f \triangleleft Y =< Y, f_{(1)} > f_{(2)}$ is the right action induced by the left coaction (9) of an element $Y \in \mathcal{U}_q(SO(N))$. Now invariance of the integral implies

$$(dx^j \triangleleft \mathcal{R}^1) < f(x) \triangleleft \mathcal{R}^2 >_t = dx^j < f(x) >_t, \tag{64}$$

because $\mathcal{R}^1 \otimes \varepsilon(\mathcal{R}^2) = 1$. Using this, (63), (52) and (51), the (rhs) of (62) becomes

$$\begin{aligned} (-1)^{N-1}\int\limits_{q^{-1}r\cdot S_q^{N-1}} \beta_{N-1}(x) q^N D^i_j dx^j &= (-1)^{N-1} D^i_j \int\limits_{r\cdot S_q^{N-1}} dx^{i_2}...dx^{i_N} f(q^{-1}x)dx^j \\ &= (-1)^{N-1} D^i_j \varepsilon^{i_2\ldots i_N j}\omega r^N < f(x) >_t \\ &= \varepsilon^{i i_2 \ldots i_N}\omega r^N < f(x) >_t \\ &= \int\limits_{r\cdot S_q^{N-1}} dx^i \beta_{N-1}(x), \end{aligned} \tag{65}$$

using (56). This shows (62), and (59) follows immediately. (60) then follows from (54).
$\square$

Another way to show (62) following an idea of Branislav Jurco [15] is to use

$$\int\limits_{r\cdot S_q^{N-1}} (\alpha_k \triangleleft SY)\beta_{N-k} = \int\limits_{r\cdot S_q^{N-1}} \alpha_k(\beta_{N-k} \triangleleft Y) \tag{66}$$

to move the action of $\mathcal{R}^2$ in (63) to the left picking up $\mathcal{R}^1 S\mathcal{R}^2$, which generates the inverse square of the antipode and thus corresponds to the D^{-1} - matrix. This approach however cannot show (28) or (45), because the commutation relations of functions are more complicated.

(58) shows in particular that the definition (55) is natural, i.e. it essentially does not matter on which side one multiplies with ω. Now we immediately obtain Stokes theorem for the integral over quantum Euclidean space, if and only if (54) holds. Noticing that $\omega(q^N Dx) = \omega(x)$, (60) implies

$$\begin{aligned}
\int_x d\alpha_{N-1}(x) &= \frac{1}{1-q}\int_x [\omega, \alpha_{N-1}]_\pm \\
&\propto \int_x \omega\alpha_{N-1} - (-1)^{N-1}\alpha_{N-1}\omega \\
&= \int_x (-1)^{N-1}\alpha_{N-1}\omega - (-1)^{N-1}\alpha_{N-1}\omega = 0 \qquad (67)
\end{aligned}$$

On the sphere, we get as easily

$$\begin{aligned}
\int\limits_{S_q^{N-1}} d\alpha_{N-2}(t) &\propto \int\limits_{S_q^{N-1}} [\omega, \alpha_{N-2}]_\pm \\
&= \omega^{-1} \int\limits_{S_q^{N-1}} \omega(\omega\alpha_{N-2} - (-1)^{N-2}\alpha_{N-2}\omega)) = 0 \qquad (68)
\end{aligned}$$

using (59) and $\omega^2 = 0$.

It is remarkable that these simple proofs only work for $q \neq 1$, nevertheless the statements reduce to the classical Stokes theorem for $q \to 1$. This shows the power of the q - deformation technique.

One can actually obtain a version of Stokes theorem with spherical boundary terms. Define

$$\int\limits_{q^k r_0}^{q^l r_0} \omega f(r) = \int\limits_{q^k r_0}^{q^l r_0} dr \frac{1}{r} f(r) = (q-1)\sum_{n=k}^{l-1} f(r_0 q^n), \qquad (69)$$

which reduces to the correct classical limit, because the (rhs) is a Riemann sum. Define

$$\int\limits_{q^k r_0 \cdot S_q^{N-1}}^{q^l r_0 \cdot S_q^{N-1}} \alpha_N(x) = \int\limits_{q^k r_0}^{q^l r_0} (\int\limits_{r\cdot S_q^{N-1}} \alpha_N(x)), \qquad (70)$$

For $l \to \infty$ and $k \to -\infty$, this becomes an integral over Euclidean space as defined before. Then

$$\int\limits_{q^k r_0 \cdot S_q^{N-1}}^{q^l r_0 \cdot S_q^{N-1}} d\alpha_{N-1} = \frac{1}{1-q}\int\limits_{q^k r_0}^{q^l r_0} (\int\limits_{r\cdot S_q^{N-1}} \omega\alpha_{N-1} - (-1)^{N-1}\alpha_{N-1}\omega)$$

$$= \frac{1}{1-q}\int_{q^k r_0}^{q^l r_0} \Big(\int_{r\cdot S_q^{N-1}} \omega\alpha_{N-1} - \int_{qr\cdot S_q^{N-1}} \omega\alpha_{N-1} \Big)$$

$$= \int_{q^l r_0\cdot S_q^{N-1}} \alpha_{N-1} - \int_{q^k r_0\cdot S_q^{N-1}} \alpha_{N-1}. \tag{71}$$

In the last line, (51), (55) and (69) was used.

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